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CONNECTION FORMULAS FOR INVARIANT EIGENDISTRIBUTIONS ON CERTAIN SEMISIMPLE SYMMETRIC SPACES

SHIGERU AOKI and SUEHIRO KATO
(青木 茂) (加藤末広)

INTRODUCTION

The harmonic analysis on hyperbolic spaces has been studied by many authors (cf. Faraut [1, p.424, Bibliographie]). The purpose of the present paper is to derive 'connection formulas' for invariant eigendistributions (I.E.D.'s) on certain hyperbolic spaces.

In the case of semisimple Lie groups, Hirai [2] gave an answer to the following problem.

Problem: Let π' be an invariant analytic function on the set G' of all regular semisimple elements in a semisimple Lie group G. Then what is a simple necessary and sufficient condition that π' defines actually (i.e., is extendable to) an I.E.D. π on G?

A kind of boundary conditions, which we call 'connection formulas', appear in the above condition. We study a similar problem for the following semisimple symmetric spaces of non-compact type: $X = U(p,q;\mathbb{F})/(U(1;\mathbb{F}) \times U(p-1,q;\mathbb{F}))$ with $p \geq 2$, where $\mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H}$.

In the above case, all I.E.D.'s on X were determined by Faraut[1]. We determine them in relation to the connection formulas by a slightly different method. An advantage of our approach is that all the I.E.D.'s supported on the nilpotent variety of X are determined in the early stage. (Faraut [1] did not mention whether such I.E.D.'s exist. Cf. Remark2.5.)

Contrary to the case of semisimple Lie groups, the I.E.D.'s are not necessarily locally summable in our case. Moreover, their restrictions to the set X' of all regular semisimple elements in X, which are analytic as in the case of semisimple Lie groups, are scarcely extendable to locally summable functions on X. These facts cause some difficulties. That is, first we have to consider the contribution of the invariant distributions with singular support(i.e.,supported on $X-X'$). In Section 2, we study these invariant distributions supported on the nilpotent variety $\mathcal{N}$ of X, in the frame work of Faraut [1]. As a by-product of this, all the I.E.D.'s supported on $\mathcal{N}$ are determined (Corollary 2.4 and Proposition 3.7). On the other hand, we need treat some divergent integrals, so we regularize them by taking their finite parts in Section 3. After these studies, we obtain our main theorem (Theorem 3.2 or Proposition 3.7). At the end of the text, we briefly compare our approach to the determination of all the I.E.D.'s, with that of Faraut [1] (using the notations of the present paper) (Remark 3.10). In Appendix, we study the invariant distributions supported on the set of singular semisimple elements.

This exposition is based upon the joint work [7] with Professor S. Sano.

1. Preliminaries.

Let $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ be the field of real numbers, complex numbers or quaternions, and put $d = \dim_{\mathbf{R}} \mathbf{F}$. We define a reductive Lie group $G = U(p,q; \mathbf{F})$ as follows:

$$G = \{g \in GL(n, \mathbf{F});\ {}^t\bar{g}I_{p,q}g = I_{p,q}\},$$

where

$$I_{p,q} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} \in GL(n, \mathbf{F})$$

and $n = p+q$. Let σ be an involutive automorphism of G defined by the formula

$$\sigma(g) = I_{1,n-1}gI_{1,n-1}.$$

By H we denote the subgroup of G consisting of all elements g ... G satisfying the condition $\sigma(g) = g$. The group H is isomorphic to $U(1; \mathbf{F}) \times U(p-1,q; \mathbf{F})$. The group G operates o...

the coset space $X = G/H$. Moreover, X is endowed with the pseudo-riemannian structure induced by the Killing form of the Lie algebra $\mathfrak{g}$ of G.

Put

$$H_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{n-2} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$j_0 = \mathbb{R}\, H_0$ and $j_1 = \mathbb{R}\, H_1$. Let θ, t be arbitrary real numbers and put $u_\theta = \exp(\theta H_0)$ and $a_t = \exp(tH_1)$, where exp stands for the exponential mapping. By simple calculations we get the following results:

$$(1.1) \quad \begin{cases} u_\theta H = u_\phi H \ \text{ if and only if } \ \theta \equiv \phi \pmod{\pi}, \\ Hu_\theta H = Hu_\phi H \ \ (-\pi/2 \leqq \theta, \phi \leqq \pi/2) \ \text{ if and only if } \ \theta = \pm\phi; \end{cases}$$

$$(1.2) \quad \begin{cases} a_t H = a_s H \ \text{ if and only if } \ t = s, \\ Ha_t H = Ha_s H \ \text{ if and only if } \ t = \pm s. \end{cases}$$

In the following, the subspaces $J_0 = \exp(j_0)H = \{u_\theta H;\ \theta \in \mathbb{R}\}$ and $J_1 = \exp(j_1)H = \{a_t H;\ t \in \mathbb{R}\}$ of the space X play a role analogous to Cartan subgroups.

We define the functions D_i on $X = G/H$ by the formula:

$$\det((t+1)\mathrm{Id}_{\mathfrak{g}} - \mathrm{Ad}_G(g\sigma(g^{-1}))) = \sum_{i=0}^{\dim G} D_i(gH)\, t^i.$$

Let k be the least natural number satisfying $D_k \not\equiv 0$. Then an element $gH \in X$ is said to be <u>regular semisimple</u> or X-<u>regular</u> if $D_k(gH) \neq 0$. We denote by X' the set of all regular semisimple elements. (Cf. Oshima-Matsuki [5, p.404].) An element $gH \in X$ is said to be <u>nilpotent</u> if $\mathrm{Ad}_G(g\sigma(g^{-1}))$ is a unipotent endomorphism of $\mathfrak{g}$. The set of all nilpotent elements is called the <u>nilpotent variety</u> of X. In our cases, it consists of two H-orbits. Put $J_i' = J_i \cap X'$ $(i=0,1)$, then we have by definition

$$J_1 - J_1' = \{H(=a_0H)\}, \qquad J_0 - J_0' = \begin{cases} \{H(=u_0H)\} & (d = 1) \\ \{H,\ u_{\pi/2}H\} & (d=2,4). \end{cases}$$

As expected, the open subset X' is the disjoint union of HJ_0' and HJ_1'. The space X is the disjoint union of $X' \cup Hu_{\pi/2}H$ and the nilpotent variety $\mathcal{N}$. And we have $\overline{HJ_i} = HJ_i \cup \mathcal{N}$ $(i = 0,1)$.

Let $\mathcal{D}(X)$ be the set of all functions of class C^∞ on X with compact support. The invariant integral F_f of a function f in $\mathcal{D}(X)$ is defined by the following formula;

(1.3) $$F_f(gH) = \int_{H/Z_H(gH)} f(hgH)\, d\dot{h},$$

where $Z_H(gH) = \{h \in H;\ hg \in gH\}$. The invariant integral F_f is a function of class C^∞ on $J_1' \cup J_0'$. For simplicity, we write often $F_f(u_\theta)$ and $F_f(a_t)$ instead of $F_f(u_\theta H)$ and $F_f(a_tH)$, respectively. Since we have $H\exp(j_0)H \cap H\exp(j_1)H = H$ and the Haar measure of $G - (H\exp(j_0)H \cup H\exp(j_1)H)$ is null, we can express the integral of $f \in \mathcal{D}(X)$ over X in terms of the invariant integral. Namely we have the following <u>Weyl's integral formula</u> (see Sano [6, p.195]):

(1.4) $$\int_X f(x)\, dx = \tau_1 \int_0^\infty F_f(a_tH)\, \Delta_1(a_tH)\, dt + \tau_0 \int_0^{\pi/2} F_f(u_\theta H)\, \Delta_0(u_\theta H)\, d\theta.$$

Here

(1.4 bis) $$\Delta_1(a_tH) = (\operatorname{sh} t)^{d(p+q-2)} (\operatorname{sh} 2t)^{d-1},$$
$$\Delta_0(u_\theta H) = (\sin \theta)^{d(p+q-2)} (\sin 2\theta)^{d-1}$$

and τ_1 and τ_0 are positive constants independent of f. Hereafter, we normalize the invariant measure appearing in (1.3) and (1.4) so that the constants τ_i are 1. We often write briefly $\Delta_0(u_\theta)$ and $\Delta_1(a_t)$ instead of $\Delta_0(u_\theta H)$ and $\Delta_1(a_tH)$, respectively. The invariant integral F_f itself is expressed as follows:

<u>Lemma</u> 1.1 (cf. Faraut [1, p.190]). Let $\mu = d(p+q-1)/2 - 1$. Then we have for $f \in \mathcal{D}(X)$,

$$F_f(u_\theta) = |\sin \theta|^{-2\mu}\, \Phi_0(\cos^2\theta) + \eta_0(\sin^2\theta)\, \Phi_1(\cos^2\theta)$$

$$F_f(a_t) = |\operatorname{sh} t|^{-2\mu}\, \Phi_0(\operatorname{ch}^2 t) + \eta_0(-\operatorname{sh}^2 t)\, \Phi_1(\operatorname{ch}^2 t)$$

for some $\varphi_0, \varphi_1 \in \mathcal{D}([0,\infty))$. Here the function η_0 is given as follows:

i) $\eta_0(\sin^2\theta) \equiv 0$, $\eta_0(-\mathrm{sh}^2 t) \equiv 1$ (p,q: odd and d = 1)

ii) $\eta_0(\sin^2\theta) \equiv 1$, $\eta_0(-\mathrm{sh}^2 t) \equiv 0$ (q: even or d = 2, 4)

iii) $\eta_0(\sin^2\theta) = \log|\sin^2\theta|$, $\eta_0(-\mathrm{sh}^2 t) = \log|\mathrm{sh}^2 t|$

(p:even, q:odd and d = 1).

Let Ω be the pseudo-Laplacian of $X = G/H$. A distribution Θ on X in the sense of Schwartz is called <u>invariant</u> if it is invariant by the subgroup H. This is written formally as

$$\Theta(h.x) = \Theta(x).$$

A distribution Θ on X is called an <u>eigendistribution</u> if

$$\Omega\,\Theta = \lambda\,\Theta$$

for some $\lambda \in \mathbb{C}$. We denote by $\mathcal{D}'_H(X)$ the set of all invariant distributions on X. The set of all invariant eigendistributions with eigenvalue λ is denoted by $\mathcal{D}'_{\lambda,H}(X)$.

Now we note that there exists a unique differential operator $\mathcal{L}^i(\Omega)$ on J'_i such that

$$(\Omega f)_{|J'_i} = \mathcal{L}^i(\Omega)(f_{|J'_i})$$

for any H-invariant function f in $C^\infty(H.J'_i)$ with $i = 0, 1$. Thus we have for any $f \in \mathcal{D}(X)$

$$F_{\Omega f}{}_{|J'_i} = \mathcal{L}^i(\Omega)\, F_f{}_{|J'_i}. \tag{1.5}$$

The differential operator $\mathcal{L}^i(\Omega)$ is called the <u>radial component</u> of Ω. Its explicit form is given as following:

$$\mathcal{L}^0(\Omega) = -\Delta_0(u_\theta)^{-1}\frac{d}{d\theta}\,\Delta_0(u_\theta)\frac{d}{d\theta} \tag{1.6}$$

$$\mathcal{L}^1(\Omega) = \Delta_1(a_t)^{-1}\frac{d}{dt}\,\Delta_1(a_t)\frac{d}{dt}. \tag{1.7}$$

The restriction to the open subset X' of any $\Theta \in \mathcal{D}'_{\lambda,H}(X)$ is a real analytic function on X'. Put $u_i = \Theta_{|J'_i}$. Then we have

$$(1.8) \qquad \varrho^{i}(\Omega)\, u_i = \lambda\, u_i .$$

Solving this differential equation, we get the following.

<u>Lemma</u> 1.2. Put $\mu = \frac{d}{2}(p+q-1)-1$, $s = \sqrt{\lambda+(\mu+\frac{d}{2})^2}$, $\alpha = \frac{1}{2}(s+\mu+\frac{d}{2})$, $\beta = \frac{1}{2}(-s+\mu+\frac{d}{2})$ and $\gamma = \frac{d}{2}$. Let u_i be the functions on J_i' satisfying the differential equations (1.8) and the conditions $u_0(u_\theta) = u_0(u_{-\theta})$ and $u_1(a_t) = u_1(a_{-t})$. Then the following holds.

i) If $\mu \in \frac{1}{2} + \mathbf{Z}$, then we have

$$u_1(a_t) = C_1 F(\alpha,\beta,\alpha+\beta-\gamma+1;\ -\mathrm{sh}^2 t)$$
$$+ C_2 |\mathrm{sh}\ t|^{-2\mu}\, F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;\ -\mathrm{sh}^2 t)$$

$$u_0(u_\theta) = C_3 F(\alpha,\beta,\alpha+\beta-\gamma+1;\ \sin^2\theta)$$
$$+ C_4 |\sin\ \theta|^{-2\mu}\, F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;\ \sin^2\theta),$$

ii) If $\mu \in \mathbf{Z}$, we have

$$u_1(a_t) = C_1 F(\alpha,\beta,\mu+1;\ -\mathrm{sh}^2 t)$$
$$+ C_2 \{F_1(\alpha,\beta,\mu+1;\ -\mathrm{sh}^2 t) + F(\alpha,\beta,\mu+1; -\mathrm{sh}^2 t)\log|\mathrm{sh}^2 t|\}$$

$$u_0(u_\theta) = C_3 F(\alpha,\beta,\mu+1;\ \sin^2\theta)$$
$$+ C_4 \{F_1(\alpha,\beta,\mu+1;\ \sin^2\theta) + F(\alpha,\beta,\mu+1;\ \sin^2\theta)\log|\sin^2\theta|\},$$

where F is the hypergeometric function which is analytic on $(-1,1)$ with $F(0) = 1$, and

$$(1.9) \qquad F_1(\alpha,\beta,\mu+1;\ z)$$
$$= (-1)^{\mu-1}\mu!\ z^{-\mu} \sum_{k=0}^{\mu-1} \frac{(-1)^k(\mu-k-1)!}{k!(\alpha-\mu+k)\cdots(\alpha-1)(\beta-\mu+k)\cdots(\beta-1)}\, z^k$$
$$+ \sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)\beta(\beta+1)\cdots(\beta+k-1)}{k!(\mu+1)\cdots\cdots(\mu+k)} \times \{\sum_{j=0}^{k-1}(\frac{1}{\alpha+j} + \frac{1}{\beta+j} - \frac{1}{j+1} - \frac{1}{\mu+1+j})\} z^k .$$

Remark 1.3. Assume that $u_0 = \Theta_{|J_0'}$ for some $\Theta \in \mathcal{D}'_{\lambda,H}(X)$. Then, $u_0(u_\theta)$ is extendable to a function analytic in θ at $\theta = \pi/2$, although $u_{\pi/2}H$ is not X-regular for $d = 2, 4$. This follows from a similar argument in the proof of Proposition 3,7 (cf. Appendix). Thus, we have the following linear relations:

i) If $\mu \in (1/2)+\mathbb{Z}$, then

$$C_3 = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \times a, \quad C_4 = \frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)} \times a \qquad (a \in \mathbb{R}),$$

ii) If $\mu \in \mathbb{Z}$, then

$$C_3 = b_0(\lambda,\mu)\ a, \qquad C_4 = \frac{(-1)^{\mu+1}\,\Gamma(\alpha+\beta-\mu)}{\Gamma(\alpha-\mu)\,\Gamma(\beta-\mu)\,\mu!} \times a \qquad (a \in \mathbb{R}),$$

where $b_0(\lambda,\mu)$ are certain constants.

2. Invariant distributions with singular support.

In this section, we study the invariant distributions supported on the nilpotent variety, introducing the space $\mathcal{H}'_\eta$. The invariant distribtions on X with support in the set of singular semisimple elements are studied in Appendix. For $f \in \mathcal{D}(X)$, set

$$Mf(\mathrm{ch}^2 t) = |\mathrm{sh}\ t|^{d(p+q-1)-2}\,(\mathrm{ch}\ t)^{d-2}\,F_f(a_t)$$

and

$$Mf(\cos^2\theta) = |\sin\theta|^{d(p+q-1)-2}\,(\cos\theta)^{d-2}\,F_f(u_\theta).$$

In view of Lemma 1.1, the functions Mf are accordant with those in Faraut [1, p.380]. Thus one has

$$Mf(\tau) = \phi_0(\tau) + \eta(\tau)\phi_1(\tau) \quad \text{for} \quad \tau \geqq 0,$$

where $\eta(\tau) = \eta_0(1-\tau)(1-\tau)^\mu$ and $\phi_i(\tau) = \tau^{d/2-1}\,\tilde{\phi}_i(\tau)$. The following lemma is due to Faraut.

Lemma 2.1 (Faraut [1, Theorem 3.1]). Let $\mathcal{H}_\eta$ denote the space of all functions Mf for $f \in \mathcal{D}(X)$. Then, with respect to a certain topology on $\mathcal{H}_\eta$, the following properties hold.

i) The mapping M of $\mathcal{D}(X)$ to $\mathcal{H}_\eta$ is continuous.

ii) Let M' denote the transposed mapping of M. Then we have

$$M'\mathcal{H}'_\eta = \mathcal{D}'_H(X).$$

iii) We have $\Omega\ M' = M'\ L$, for $L = a(\tau)\frac{d^2}{d\tau^2} + b(\tau)\frac{d}{d\tau}$, where $a(\tau) = 4\tau(\tau-1)$, $b(\tau) = 4\{(d/2+\mu+1)\tau-d/2\}$.

<u>Remark</u> 2.2. We define functionals A_k, B_k $(k=0,1,2,\cdots)$ by the asymptotic expansion $\phi(\tau) \sim \sum_{k=0}^{\infty} A_k(\phi)(1-\tau)^k + \eta(\tau)\sum_{k=0}^{\infty} B_k(\phi)(1-\tau)^k$. Then we have $A_k, B_k \in \mathcal{H}'_\eta$. Moreover, the subspace of $\mathcal{H}'_\eta$ consisting of finite linear combinations of A_k's and B_k's is precisely the orthogonal complement of $\mathcal{D}([0,1)\cup(1,\infty))$. Henceforth, we denote by $\tilde{A}_k$ the distribution $M'A_k \in \mathcal{D}'_H(X)$. Similarly $M'B_k$ is denoted by $\tilde{B}_k$.

<u>Lemma</u> 2.3. Let Ω be the pseudo-Laplacian of X. Put $\tilde{A}_k = 0$ and $\tilde{B}_k = 0$ for negative integers k. Then the invariant distributions $\tilde{A}_k$'s and $\tilde{B}_k$'s satisfy the following equalities:

$$(2.1)\begin{cases} \Omega\tilde{A}_k = 4(k+1)\ \{(k+1-\mu-2/d)\tilde{A}_K - (k+1-\mu)\tilde{A}_{k+1}\} \\ \qquad + 4\{2k-\mu+2-d/2)\tilde{B}_{k-\mu} - (2k-\mu+2)\tilde{B}_{k-\mu+1}\} \\ \qquad\qquad \text{(for } d = 1,\ p\text{: even and } q\text{: odd)} \\ \Omega\tilde{A}_k = 4(k+1)\{(k+1-\mu-d/2)\tilde{A}_k-(k+1-\mu)\tilde{A}_{k+1}\} \\ \qquad\qquad \text{(otherwise)}, \end{cases}$$

$$(2.2)\quad \Omega\tilde{B}_k = 4(\mu+k+1)\ \{(k+1-d/2)\tilde{B}_k - (k+1)\tilde{B}_{k+1}\} \qquad \text{(for all } d,\ p \text{ and } q).$$

Proof. Let $\phi = \phi_0 + \eta\, \phi_1$, and let $\phi_0(\tau) \sim \sum_{k=0}^{\infty} A_k(\phi)\,(1-\tau)^k$ and $\phi_1(\tau) \sim \sum_{k=0}^{\infty} B_k(\phi)\,(1-\tau)^k$ be asymptotic expansions of ϕ_0 and ϕ_1, respectively. We note that

$$L^* = \frac{d^2}{d\tau^2}\, a(\tau) - \frac{d^2}{d\tau^2}\, b(\tau)$$

$$= 4\,\frac{d^2}{d\tau^2}\,\{(1-\tau)^2 - (1-\tau)\} + 4\,\frac{d}{d\tau}\,\{(d/2+\mu+1)(1-\tau) - (\mu+1)\}.$$

Hence it follows easily that

$$L^*\phi_0(\tau) \sim \sum_{k=0}^{\infty} 4(k+1)\{(k+1-\mu-d/2)A_k(\phi)+(k+1-\mu)A_{k+1}(\phi)\}\,(1-\tau)^k,$$

$$L^*((1-\tau)^{\mu}\phi_1(\tau)) \sim \sum_{k=0}^{\infty} 4(\mu+k+1)\,\{(k+1-d/2)B_k(\phi) - (k+1)B_{k+1}(\phi)\}$$
$$\times\,(1-\tau)^{k+\mu}.$$

On the other hand, we have $L^*(\eta\phi_1)(\tau) = \eta_0(1-\tau)L^*((1-\tau)^{\mu}\phi_1(\tau)) + R(\tau)$, where

$$R(\tau) \sim \sum_{k=0}^{\infty} 4\{(2k-\mu+2-d/2)B_{k-\mu} - (2k-\mu+2)B_{k-\mu+1}\}\,(1-\tau)^k \quad \text{(for } d = 1,\ p\text{: even and } q\text{: odd)}$$
$$= 0 \quad \text{(otherwise)}.$$

In view of Lemma 2.1 iii), it is now straightforward to prove Lemma 2.3. q.e.d.

Noting the equality (2.1) (and Corollary A.4 in Appendix), we obtain the following corollary that shows the existence of (non trivial) invariant eigendistributions on X with singular support.

<u>Corollary</u> 2.4. Put $\Lambda_{d,\mu} = \{\lambda;\ \lambda = 4(k+1)(k+1-\mu-d/2) \quad (0 \leqq k \leqq \mu-1)\} = \{\lambda(s);\ s = \pm(\rho+2r)\ (r = -1, -2, \cdots, -\mu)\}$, with $\lambda(s) =$

$s^2-\rho^2$ and $\rho = \mu+d/2$. Then, in the case of [d = 1, p:odd, q:even] or [d = 2, 4], there are invariant eigendistributions supported on the nilpotent variety with eigenvalue λ, for any $\lambda \in \Lambda_{d,\mu}$. Otherwise, there are no invariant eigendistributions with singular support.

Remark 2.5. After writing the paper [7], the authers were informed that the fact comformable with the above corollary was announced in Kengmana [3, Remark 1)] and proved in his thesis (May, 1984; Harvard University).

3. Main theorem.

Let $u_0(u_\theta)$ and $u_1(a_t)$ be functions on J_0' and J_1', respectively, satisfying the conditions:

i) $\mathfrak{Y}^i(\Omega)\, u_i = \lambda\, u_i$ for some $\lambda \in \mathbb{C}$ $(i = 0, 1)$

ii) $u_0(u_\theta) = u_0(u_{-\theta})$ and $u_1(a_t) = u_1(a_{-t})$.

Then the integrals:

$$\int_0^\infty u_1(a_t)\, F_f(a_t)\, \Delta_1(a_t)\, dt \tag{3.1}$$

and

$$\int_0^{\pi/2} u_0(u_\theta)\, F_f(u_\theta)\, \Delta_0(u_\theta)\, d\theta \tag{3.2}$$

are not convergent in general. (By Lemma A.2 in Appendix, the integrals (3.2) are always convergent around $\theta = \pi/2$.) Nevertheless, the following lemma can be proved with the use of Lemma 1.1 and Lemma 1.2.

Lemma 3.1. For functions $u_0(u_\theta)$ and $u_1(a_t)$ satisfying the above conditions i) and ii), we have

$$\int_\delta^\infty u_1(a_t)\, F_f(a_t)\, \Delta_1(a_t)\, dt \tag{3.3}$$

$$= \sum_{i=-1}^{-2\mu+1} \alpha_i (\mathrm{sh}\ \delta)^i + \sum_{i=-1}^{-2\mu+1} \beta_i (\mathrm{sh}\ \delta)^i \log(\mathrm{sh}\ \delta)$$

$$+ \gamma(\log(\mathrm{sh}\ \delta))^2 + G(\mathrm{sh}\ \delta)$$

and

$$\int_\delta^{\pi/2} u_0(u_\theta)\ F_f(u_\theta)\ \Delta_0(u_\theta)\ d\theta$$

(3.4)
$$= \sum_{i=-1}^{-2\mu+1} \alpha_i' (\sin \delta)^i + \sum_{i=-1}^{-2\mu+1} \beta_i' (\sin \delta)^i \log(\sin \delta)$$

$$+ \gamma' (\log(\sin \delta))^2 + G'(\sin \delta)$$

for $\delta > 0$. Here α_i, β_i, γ_i, α_i', β_i', $\gamma_i' \in \mathbb{C}$ and G, G' are the functions such that $G(0) = \lim_{\delta \to +0} G(\mathrm{sh}\ \delta)$ and $G'(0) = \lim_{\delta \to +0} G'(\sin \delta)$ exist.

We set

(3.5)
$$\mathrm{P.f.} \int_0^\infty u_1(a_t)\ F_f(a_t)\ \Delta_1(a_t)\ dt = G(0)$$

and

(3.6)
$$\mathrm{P.f.} \int_0^{\pi/2} u_0(u_\theta)\ F_f(u_\theta)\ \Delta_0(u_\theta)\ d\theta = G'(0),$$

which are called finite parts of the integrals (3.1) and (3.2). Thus, in terms of the functions u_0 and u_1, one can define a functional $\Theta = \Theta_{u_0, u_1, (\alpha_k), (\beta_k)}$ on $\mathfrak{D}(X)$ by

(3.7)
$$\langle \Theta, f \rangle = \mathrm{P.f.} \int_0^\infty u_1(a_t)\ F_f(a_t)\ \Delta_1(a_t)\ dt$$

$$+ \mathrm{P.f.} \int_0^{\pi/2} u_0(u_\theta)\ F_f(u_\theta)\ \Delta_0(u_\theta)\ d\theta$$

$$+ \sum_{k=0}^{\infty} \alpha_k\ \tilde{A}_k(f) + \sum_{k=0}^{\infty} \beta_k\ \tilde{B}_k(f)$$

for $f \in \mathfrak{D}(X)$. Here the third and fourth terms of the right hand side are finite sums. We write $\Theta_{u_0, u_1} = \Theta_{u_0, u_1, (\alpha_k), (\beta_k)}$ in the case of $\alpha_k = \beta_k = 0$ for all k. As is seen easily by Faraut [1, Theorem 3.1], Θ is an invariant distribution on X. Similarly, one can define an invariant distribution $\Theta' = \Theta'_{u_0,u_1}$ on the set X' by

$$(3.7)' \qquad \langle \Theta', f \rangle = \int_0^\infty u_1(a_t)\, F_f(a_t)\, \Delta_1(a_t)\, dt + \int_0^{\pi/2} u_0(u_\theta)\, F_f(u_\theta)\, \Delta_0(u_\theta)\, d\theta$$

for $f \in \mathcal{D}(X')$.

Now we shall state our main theorem.

<u>Theorem</u> 3.2. Let $u_0(u_\theta)$ and $u_1(a_t)$ be functions on J'_0 and J'_1, respectively, satisfying the following three conditions:

i) $\rho^i(\Omega)\, u_i = \lambda\, u_i$ for some $\lambda \in \mathbf{C}$ $(i = 0, 1)$.

ii) $u_0(u_\theta) = u_0(u_{-\theta})$ and $u_1(a_t) = u_1(a_{-t})$. (Note that $u_0(u_{\theta+\pi}) = u_0(u_\theta)$ always holds in view of (1.1).)

iii) In the case of $d = 2, 4$, $u_0(u_\theta)$ is extendable to a function analytic in θ at $\theta = \pi/2$.

Let Θ'_{u_0,u_1} be the invariant distribution on the set X' defined by (3.7)′. Then, a necessary and sufficient condition in order that Θ'_{u_0,u_1} is extendable to an invariant eigendistribution on X is the following:

Case 1: $d = 1$, p, q, odd.

$$(3.8) \qquad \lim_{t \to +0} H_1(\Delta_1 u_1)(a_t) = 0$$

and

$$(3.9) \qquad \lim_{t \to +0} \{H_1^{2\mu+1}(\Delta_1 u_1)(a_t) - H_0^{2\mu+1}(\Delta_0 u_0)(u_t)\} = 0.$$

Case 2: $d = 1$, p, q, even.

$$(3.10) \qquad \lim_{\theta \to +0} H_0(\Delta_0 u_0)(u_\theta) = 0$$

and (3.9).

Case 3: $d = 1$, p, even, q, odd.

(3.11) $$\lim_{t\to+0} H_1(\Delta_1 u_1)(a_t) = \lim_{\theta\to+0} H_0(\Delta_0 u_0)(u_\theta)$$

and (3.9).

Case 4 a): [$d = 1$, p, odd, q, even] or [$d = 2, 4$],

$\lambda \notin \Lambda_{d,\mu}$.

The equalities (3.8) and (3.10).

Case 4 b): [$d = 1$, p, odd, q, even] or [$d = 2, 4$],

$\lambda \in \Lambda_{d,\mu}$.

The equalities (3.8), (3.9) and (3.10).

[Note that the set $\Lambda_{d,\mu}$ has been defined in Corollary 2.4.]

The above conditions in Theorem 3.2 are called <u>connection formulas</u> for $u_0(u_\theta)$ and $u_1(a_t)$.

<u>Remark</u> 3.3. Let u_0 and u_1 be functions on J_0' and J_1' respectively satisfying the conditions i) and ii). Then u_0 satisfies the condition iii), whenever Θ'_{u_0,u_1} is extendable to an element of $\mathcal{D}'_{\lambda,H}(X)$ (cf. Proposition A.1 in Appendix).

<u>Remark</u> 3.4. In the case where there exist no invariant eigendistributions on X with singular support with eigenvalue λ, the distribution Θ'_{u_0,u_1} in Theorem 3.2 is uniquely extendable to an invariant eigendistribution on X.

In the following, we shall prove the above theorem by a series of lemmas.

<u>Lemma</u> 3.5 For $\delta > 0$, it follows that

$$\int_\delta^\infty u_1(a_t)\ \{\mathfrak{L}^1(\Omega)F_f(a_t)\}\ \Delta_1(a_t)\ dt$$

$$= \int_{\delta}^{\infty} \{\Upsilon^1(\Omega)u_1(a_t)\}\, F_f(a_t)\, \Delta_1(a_t)\, dt$$

$$- \Delta_1(a_t)\, [\{\tfrac{d}{dt}\, u_1(a_t)\}\, F_f(a_t) - u_1(a_t)\, \tfrac{d}{dt}\, F_f(a_t)]\, |_{t=\delta}$$

and

$$\int_{\delta}^{\pi/2} u_0(u_\theta)\, \{\Upsilon^0(\Omega)F_f(u_\theta)\}\, \Delta_0(u_\theta)\, d\theta$$

$$= \int_{\delta}^{\pi/2} \{\Upsilon^0(\Omega)u_0(u_\theta)\}\, F_f(u_\theta)\, \Delta_0(u_\theta)\, d\theta$$

$$+ \Delta_0(u_\theta)\, [\{\tfrac{d}{d\theta}\, u_0(u_\theta)\}\, F_f(u_\theta) - u_0(u_\theta)\tfrac{d}{d\theta}\, F_f(u_\theta)]\, |_{\theta=\delta}.$$

Proof. Notice that $F_f(a_t) = 0$ for a sufficient large number t. On the other hand, since (an extension of) $u_0(u_\theta)$ is analytic in θ at $\theta = \pi/2$ and u_0 and F_f are H-invariant, we have

$$\Delta_0(u_\theta)[\{\tfrac{d}{d\theta}\, u_0(u_\theta)\}\, F_f(u_\theta) - u_0(u_\theta)\, \tfrac{d}{d\theta}\, F_f(u_\theta)]\, |_{\theta=\pi/2} = 0.$$

Thus the lemma follows from (1.6), (1.7) and integration by parts.

q.e.d.

Lemma 3.6. Keep the assumptions of the theorem. Then for each invariant distribution $\Theta = \Theta_{u_0,u_1}$, one obtains the following equalities.

$(\Omega-\lambda)\Theta$

$= (C_1-C_3)\, 4\mu\, \tilde{A}_0 - C_2\, 4\mu\, \tilde{B}_0$ (Case 1)

$= (C_1-C_3)\, 4\mu\tilde{A}_0 + C_4 4\mu\, \tilde{B}_0$ (Case 2)

$= (C_4-C_2)4[\sum_{k=0}^{\mu-1} \{(\mu-2k-2+d/2)\tilde{f}_{\mu-k-1} + (-\mu+2k)\tilde{f}_{\mu-k}\}\, \tilde{A}_k + \mu\tilde{f}_0\, \tilde{A}_\mu]$

$+(C_4-C_2)4(\tilde{f}_0\tilde{B}_0-\tilde{A}_0) + (C_1-C_3)4\mu\tilde{A}_0$

(Case 3, $\mu \neq 0$)

$= (C_4-C_2)4(\tilde{f}_0\tilde{B}_0-\tilde{A}_0) + (C_3-C_1)4\tilde{B}_0$

(Case 3, $\mu = 0$, hence $d = 1$, $p = 2$, $q = 1$)

$$= (C_4 - C_2)4 \sum_{k=0}^{\mu-1} (\mu-2k-2+d/2)\, \tilde{f}_{\mu-1-k}\, \tilde{A}_k$$

$$+ (C_4-C_2)\ 4 \sum_{k=0}^{\mu} (2k-\mu)\tilde{f}_{\mu-k}\tilde{A}_k$$

$$+ (C_2-C_4)\ 4\tilde{A}_0 + (C_1-C_3)4\mu\ \tilde{A}_0 + C_4 4\mu\tilde{f}_0\ \tilde{B}_0 \qquad \text{(Case 4)},$$

where C_1, C_2, C_3 and C_4 are the coefficients as in Lemma 1.2, and $\tilde{f}_k$ is the coefficient of $z^{k-\mu}$ of the function in (1.9).

Proof. We suppose that $d = 1$ and that p, q: even (Case 2). In view of Lemma 1.1 and Lemma 1.2, we have for $t, \theta > 0$

(3.12) $\Delta_1(a_t)\ [\{\frac{d}{dt}u_1(a_t)\}\ F_f(a_t) - u_1(a_t)\ \frac{d}{dt}F_f(a_t)]$

$$=[C_1\ \frac{d}{dt}F(-sh^2t) + C_2\ \{\frac{d}{dt}|sh\ t|^{-2\mu}\ \tilde{F}(-sh^2t)\}]\ |sh\ t|\ \phi_0(1+sh^2t)$$

$$-\{C_1F(-sh^2t) + C_2|sh\ t|^{-2\mu}\ \tilde{F}(-sh^2t)\}(sh\ t)^{2\mu+1}$$

$$\times\frac{d}{dt}\{|sh^2t|^{-\mu}\phi_0(1+sh^2t)\}$$

and

(3.13) $\Delta_0(u_\theta)\ [\{\frac{d}{d\theta}u_0(u_\theta)\}\ F_f(u_\theta) - u_0(u_\theta)\ \frac{d}{d\theta}F_f(u_\theta)]$

$$=[C_3\ \frac{d}{d\theta}F(\sin^2\theta) + C_4\ \{\frac{d}{d\theta}|\sin\ \theta|^{-2\mu}\ \tilde{F}(\sin^2\theta)\}]\ |\sin\ \theta|\ \phi_0(1-\sin^2\theta)$$

$$+[C_3\ \frac{d}{d\theta}\ F(\sin^2\theta)+C_4\ \frac{d}{dt}\{(\sin\ \theta)^{-2\mu}\ \tilde{F}(\sin^2\theta)\}](\sin\ \theta)^{2\mu+1}$$

$$\times\phi_1(1-\sin^2\theta)$$

$$-\{C_3F(\sin^2\theta)+C_4(\sin\ \theta)^{-2\mu}\ \tilde{F}(\sin^2\theta)\}|\sin\ \theta|^{2\mu+1}\ \frac{d}{d\theta}\{(\sin\ \theta)^{-2\mu}$$

$$\times\ \phi_0(1-\sin^2\theta)\}$$

$$-\{C_3F(\sin^2\theta) + C_4(\sin\ \theta)^{-2\mu}\ \tilde{F}(\sin^2\theta)\}\ |\sin\ \theta|^{2\mu+1}\ \frac{d}{d\theta}\phi_1(1-\sin^2\theta),$$

where $F(z) = F(\alpha,\beta,\alpha+\beta-\gamma+1;\ z)$ and $\tilde{F}(z) = F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;\ z)$. We expand (3.12) (resp. (3.13)) with respect to $-sh^2t$ (resp.

$\sin^2\theta$) and calculate the constant term. Thus our lemma follows from the definition of Θ and Lemma 3.5. One can verify the assertion of the lemma similarly in the other cases.

q.e.d.

The following proposition follows from Lemma 3.6 and Lemma 2.3.

<u>Proposition</u> 3.7. Keep the assumptions of the theorem. Then $\Theta = \Theta_{u_0,u_1,(\alpha_k),(\beta_k)}$ is an eigendistribution on X with eigenvalue λ if and only if

$C_2 = 0,\quad C_1 = C_3$ and $\alpha_k,\ \beta_k = 0$ (Case 1)

$C_4 = 0,\quad C_1 = C_3$ and $\alpha_k,\ \beta_k = 0$ (Case 2)

$C_2 = C_4,\quad C_1 = C_3$ and $\alpha_k,\ \beta_k = 0$ (Case 3)

$C_2 = C_4 = 0,\quad C_1 = C_3 + [\{4(\mu+d/2-1)+\lambda\}/4\mu]\ \alpha_0$ and

$$\alpha_k = \frac{4(k+2)(\mu+d/2-k-2)+\lambda}{4(k+1)(\mu-k-1)}\ \alpha_{k+1} \quad \text{with } 0 \leqq k \leqq \mu-2$$

(Case 4),

where C_1, C_2, C_3 and C_4 are the coefficients as in Lemma 1.2, and α_k, β_k are the numbers in (3.7). [In virtue of the fact that u_0 is (extendable to) a function analytic in θ at $\theta = \pi/2$, there is a certain linear relation between C_3 and C_4 (see Remark 1.3).]

We note that $\tilde{B}_k$'s are supported at the origin of $X = G/H$ and that $\tilde{A}_k$'s are supported on the nilpotent variety $\mathcal{N}$ of X. As is seen in the above proposition, $\tilde{B}_k$'s never appear and $\tilde{A}_k$'s appear only in the case of [d = 1, p: odd, q; even] or [d = 2, 4] (Case 4).

<u>Lemma</u> 3.8. Under the same assumption as in the theorem, it follows that

i) $C_2 = \lim_{t\to+0} H_1\ (\Delta_1\ u_1)(a_t)$ and $C_4 = \lim_{\theta\to+0} H_0\ (\Delta_0\ u_0)(u_\theta)$,

ii) We assume, moreover, that $\mu \in (1/2)+\mathbf{Z}$ or that $C_2 = C_4$ in the case where μ is an integer. Then, the condition $C_1 = C_3$ holds if and only if

$$(3.9) \qquad \lim_{t\to+0} \{H_1^{2\mu+1}(\Delta_1 u_1)(u_t) - H_0^{2\mu+1}(\Delta_0 u_0)(a_t)\} = 0$$

iii) In the case that $\nu \in (1/2)+\mathbf{Z}$, (3.9) is equivarent to the condition

$$(3.9)' \qquad \lim_{\theta\to+0} H_0^{2\mu+1}(\Delta_0 u_0)(u_\theta) = \lim_{t\to+0} H_1^{2\mu+1}(\Delta_1 u_1)(a_t)$$

Combining Proposition 3.7 with Lemma 3.8, we complete the proof of Theorem 3.2.

As a corollary to Proposition 3.7, one can show the following.

<u>Corollary</u> 3.9 (Faraut [1, Theorem 3.2]). i) If $d = 1$ and q is odd, then for all $\lambda \in \mathbf{C}$

$$\dim \mathcal{D}'_{\lambda,H}(X) = 1.$$

ii) If [$d = 1$ and q is even] or if [$d = 2$ or 4](case 2,case4), then for $\lambda \neq 2r\{2r+d(p+q)-2\}$, $r = 0,1,2,\cdots$ (i.e., $\lambda \notin \{\lambda(s);\ s = \pm(\rho+2r),\ r = 0,1,2,\cdots\}$),

$$\dim \mathcal{D}'_{\lambda,H}(X) = 1,$$

and for $\lambda = 2r\{2r+d(p+q)-2\}$, $r = 0,1,2,\cdots$,

$$\dim \mathcal{D}'_{\lambda,H}(X) = 2.$$

In the above corollary, whether $\dim \mathcal{D}'_{\lambda,H}(X) = 1$ or 2, is determined by the linear relation between C_3 and C_4 in Proposition 3.7, (cf. Remark 1.3).

<u>Remark</u> 3.10. In this remark, we shall first summarize how Faraut [1] derived Corollary 3.9, dividing into several paragraphs.

0) In view of Lemma 2.1, one has only to consider the equation

$$(*) \qquad LS = \lambda S$$

in $\mathcal{H}'_\eta$.

1) To investigate the differential equation (*) around $\tau = 1$, put $\mathcal{H}_\eta((0,\infty)) = \{\phi \in \mathcal{H}_\eta;\ \mathrm{supp}\ \phi \subset (0,\infty)\}$ and $\mathcal{H}_\eta([0,1)) = \{\phi \in \mathcal{H}_\eta;\ \mathrm{supp}\ \phi \subset [0,1)\}$. The dual space $\mathcal{H}'_\eta((0,\infty))$ of the former corresponds to the space of all H-invariant distribution on $X - Hu_{\pi/2}H$ and that of the latter corresponds to that on $H.J'_0 \cup H.u_{\pi/2}H$.

2) As is known, the solution of (*) in $\mathcal{D}'((0,\infty))$ is 3-dimensional. One gives a basis $\langle S_0, S_1, S_2\rangle$ of this vector space explicitely so that each S_i is naturally extended to an element of $\mathcal{H}'_\eta((0,\infty))$.

3) Faraut [1, Appendix] showed that two of $\langle S_0, S_1, S_2\rangle$ is the solution of (*) and that these two solutions constitute a basis of the space of all the solutions. These facts are deduced by verifying that

$$LS_i - \lambda S_i = \begin{cases} cB_0 \quad (c \neq 0) & \text{for one } S_i \\ 0 & \text{for the others} \end{cases}$$

and

$$LB_k = 4(k+1)(k+\mu+1)B_{k+1} + \sum_{j=0}^{k} \beta_{kj}B_j \qquad (\beta_{kj} \in \mathbf{C})$$

(cf. Lemma 2.3 (2.2) in this paper), since the complement of $\mathcal{D}((0,\infty))$ in $\mathcal{H}'_\eta((0,\infty))$ is spanned by B_k's.

4) The solution of (*) in $\mathcal{H}'_\eta$ must be regular at $\tau = 0$. Hence one gets the 1-dimensional space of the solutions of (*) in $\mathcal{H}'_\eta([0,1))$. Seeking the continuation of a solution in $\mathcal{H}'_\eta([0,1))$ tc solutions in $\mathcal{H}'_\eta((0,\infty))$, which has been obtained in 3), one gets Corollary 3.9.

On the other hand, our program to find all I.E.D.'s on X is more simple. We shall summarize it, taking as example the case tha F = C or H. We recall that

$$X = (Hu_{\pi/2}H \cup HJ'_0) \cup \mathcal{N} \cup HJ'_1 \qquad \text{(cf. §1).}$$

Let $u_0 = u_{0;C_3,C_4}$, $u_1 = u_{1;C_1,C_2}$ be the solutions of $\mathfrak{L}^i(\Omega)\ u_i$

$= \lambda u_i$ defined in Lemma 1.2. We extend the corresponding invariant distribution on $H.J_i'$ to an invariant distribution $P.f.\tilde{} u_i$ on X supported on the closure $\overline{H.J_i'}$ of $H.J_i'$. $\tilde{A}_k$'s, $\tilde{B}_k$'s and $\tilde{E}_k$'s are defined so that $\{ \tilde{A}_k , \tilde{B}_k ; k = 0,1,2,\ldots\}$ is a basis of $\{\Theta \in \mathcal{D}_H'(X);\ \mathrm{supp}\,\Theta \subset \mathcal{N}\}$ and $\{\tilde{E}_k;\ k = 0,1,2,\ldots\}$ is a basis of $\{\Theta \in \mathcal{D}_H'(X);\ \mathrm{supp}\,\Theta \subset H.u_{\pi/2}H\}$.

Any element $\Theta \in \mathcal{D}_{\lambda,H}'(X)$ is expressed as the form:

$$\Theta = P.f.\tilde{} u_{1;C_1,C_2} + P.f.\tilde{} u_{0;C_3,C_4} + \sum_k (\alpha_k\tilde{A}_k + \beta_k\tilde{B}_k) + \sum_k \tau_k\tilde{E}_k.$$

Hence, calculating $\Omega(P.f.\tilde{} u_{1;C_1,C_2})$, $\Omega(P.f.\tilde{} u_{0;C_3,C_4})$, $\Omega\tilde{A}_k$, $\Omega\tilde{B}_k$ and $\Omega\tilde{E}_k$ explicitly (Lemmmas 3.6, 2.3 and (A.3)), we determine all the invariant eigendistributions on X. In carrying out the above program, we utilize some tecniques used in Faraut[1](e.g. 0)).

As we have seen above, the point of our method is to decompose $\Theta \in \mathcal{D}_H'(X)$ into invariant distributions with support in various subsets (i.e. $\overline{H.J_i}, \mathcal{N}$ & $H.u_{\frac{\pi}{2}}H$). In particular, investigating the contribution of A_k's and B_k's directly, we can determine the invariant eigendistributions supported on nilpotent variety.

Appendix

In this appendix, we shall prove the following assertion stated in Remark 1.3.

<u>Proposition</u> A.1. Let u_0 be the restriction to J_0' of an invariant eigendistribution Θ on X with eigenvalue λ. Then $u_0(u_\theta)$ is extendable to a function analytic in θ at $\theta = \pi/2$.

Since $u_{\pi/2}H$ is a regular semisimple element for $d = 1$, the above assertion is trivial in this case. Thus we assume that $d = 2, 4$ in the following. First we recall that $F_f(u_\theta)$ is of the form:

(A.1) $$F_f(u_\theta) = \psi(\cos^2\theta)$$

for some $\psi(\tau) \in \mathcal{B}([0,\infty))$ (Lemma 1.1). On the other hand, since u_0 satisfies the differential equation $\mathfrak{L}^0(\Omega)u_0 = \lambda u_0$ by the assumption, u_0 can be expressed as

(A.2) $$\begin{aligned} u_0(u_\theta) = {} & C_5\, F(\alpha,\beta,d/2;\, \cos^2\theta) \\ & + C_6\{(\cos^2\theta)^{1-(d/2)}\, \tilde{F}_d(\alpha,\beta,d/2;\, \cos^2\theta) \\ & + F(\alpha,\beta,d/2;\, \cos^2\theta)\, \log(\cos^2\theta)\} \end{aligned}$$

where $\tilde{F}_d(\alpha,\beta,d/2;\, \tau)$ are certain analytic functions in $\tau \in [0, 1)$ with $\tilde{F}_d(\alpha,\beta,d/2;\, 0) \neq 0$.

The following lemma follows from (A.1) and (A.2) immediately.

<u>Lemma</u> A.2. The integral:

$$\int_\delta^{\pi/2} u_0(u_\theta)\, F_f(u_\theta)\, \Delta_0(u_\theta)\, d\theta$$

converges absolutely for $0 < \delta \leqq \pi/2$.

Now we consider any invariant distribution $\tilde{E}$ with support in $Hu_{\pi/2}H$. By $\tilde{E}_k$ we denote the invariant distribution on X defined by the asymptotic expansion:

(A.1)′ $$F_f(u_\theta) \sim \sum_{k=0}^{\infty} \tilde{E}_k(f)\, (\cos^2\theta)^k \qquad (f \in \mathcal{D}(X))$$

(cf. (A.1)). Then we see that $\tilde{E}$ can be expressed as a finite linear combination of $\tilde{E}_k$'s; i.e.,

$$\tilde{E}(f) = \sum_{k=0}^{\infty} \tau_k\, \tilde{E}_k(f) \qquad \text{(finite sum)}$$

for $f \in \mathcal{D}(X)$.

Let $T = T_{u_0,(\tau_k)}$ be an invariant distribution on X defined by

$$\langle T_{u_0,(\tau_k)}, f\rangle = \int_{\pi/4}^{\pi/2} u_0(u_\theta)\, F_f(u_\theta)\, \Delta_0(u_\theta)\, d\theta + \sum_{k=0}^{\infty} \tau_k\, \tilde{E}_k(f)$$

for $f \in \mathfrak{D}(X)$. If all τ_k are zero, we abbreviate T_{u_0} for $T_{u_0,(\tau_k)}$. Owing the above facts, to prove Proposition A.1, it is sufficient to show the following:

Lemma A.3. Under the assumptions of Proposition A.1, an invariant distribution $T = T_{u_0,(\tau_k)}$ satisfies the condition:

$$\langle \Omega T, f\rangle = \lambda\langle T, f\rangle$$

for $f \in \mathfrak{D}_0(X)$, if and only if $C_6 = \tau_k = 0$ for all k. Here u_0 is of the form (A.2) and

$$\mathfrak{D}_0(X) = \{f \in \mathfrak{D}(X);\ F_f(u_\theta) = \psi(\cos^2\theta),\quad \operatorname{supp}\psi \subset [0, 1/2)\}.$$

Proof. First, by a similar method to the proof of Lemma 2.3, we get

$$\Omega\tilde{E}_k = 4\{(d/2)+k\}\{(k-\mu)\tilde{E}_k-(k+1)\tilde{E}_{k+1}\} \tag{A.3}$$

with $0 \leqq k \leqq \infty$. On the other hand, it follows from integration by parts, the assumption $\mathfrak{L}^0(\Omega)u_0 = \lambda u_0$, (A.1)′ and (A.2) that

$$\begin{aligned}
&\langle \Omega T, f\rangle - \langle T, f\rangle \\
&= \Delta_0(u_\theta)\ \{u_0(u_\theta)\ \tfrac{d}{d\theta}F_f(u_\theta) - (\tfrac{d}{d\theta}u_0(u_\theta))F_f(u_\theta)\}\ |_{\theta=\pi/2} \\
&= \begin{cases} -4\ C_6\ \tilde{E}_0(f) & (d = 2) \\ 4\ C_6\ \tilde{F}_d(\alpha,\beta,d/2;\ 0)\ \tilde{E}_0(f) & (d = 4) \end{cases}
\end{aligned} \tag{A.4}$$

for $f \in \mathfrak{D}_0(X)$. Now our lemma follows from (A.3) and (A.4) immediately. q.e.d.

In view of the proof of Lemma A.3, we find the following:

Corollary A.4. There are no invariant eigendistributions with support in $Hu_{\pi/2}H$.

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Shigeru Aoki
Department of Mathematics
University of Tokyo
Hongo, Tokyo 113, Japan

Suehiro Kato
Department of Mathematics
Rikkio University
Tokyo 117, Japan